Crisis-induced intermittency in truncated mean field dynamos

Eurico Covas* and Reza Tavakol†
Astronomy Unit, School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, United Kingdom
(Received 1 October 1996)

We investigate the detailed dynamics of a truncated $\alpha\omega$ dynamo model with a dynamic α effect. We find the presence of multiple attractors, including two chaotic attractors with a fractal basin boundary that merge to form a single attractor as the control parameter is increased. By considering phase portraits and the scaling of averaged times of transitions between the two attractors, we demonstrate that this merging is accompanied by a crisis-induced intermittency. We also find a range of parameter values over which the system has a fractal parameter dependence for fixed initial conditions. To the authors' knowledge, this is the first time this type of intermittency has been observed in a dynamo model and it could be of potential importance in accounting for some forms of intermittency in the solar and stellar output. [S1063-651X(97)05106-4]

PACS number(s): 05.45.+b

I. INTRODUCTION

Intermittent-type behavior has been observed in a wide range of experimental and numerical studies of dynamical systems. Theoretical attempts at understanding such modes of behavior fall into two groups: (i) stochastic, involving models in which intermittency is brought about through the presence of some form of external noise, and (ii) deterministic, where the mechanism of production of intermittency is purely internal.

Here we concentrate on the latter and in particular on an important subset of such mechanisms referred to as ''crisis intermittency'' [1,2], whereby attractors underlying the dynamics change suddenly as a system parameter is varied. There are both experimental and numerical evidence for such modes of behavior (see, for example, [3,2,4–6] and references therein). As far as their detailed underlying mechanism and temporal signature are concerned, crises come in three varieties [2]. Of particular interest for our discussion here is the type of intermittency (which can occur in systems with symmetry) referred to as ''attractor merging crisis,'' whereby as a system parameter is varied, two or more chaotic attractors merge to form a single attractor.

An important potential domain of relevance of dynamical intermittency is in understanding the mechanism of production of the so-called grand or Maunder-type minima in solar and stellar activity, during which the amplitude of the stellar cycle is greatly diminished [7]. Many attempts have recently been made to account for such a behavior by employing various classes of models, including truncated models involving ordinary differential equations (ODEs) (cf. [8–10]) as well as axisymmetric mean-field dynamo models modeled on partial differential equations, in both spherical shell [11–14] and torus [15] topologies. In order to transcend phenomenological explanations and establish the underlying mechanism for such behavior (or behaviors, since after all more than one intermittency mechanism may occur even in a single model but at different system parameters), it is of vital importance to be able to distinguish between the various intermittency mechanisms and this in turn is greatly assisted by determining the forms of intermittency that can occur for stellar dynamo models.

Here we consider a truncation of an axisymmetric mean-field dynamo model and demonstrate that it can possess crisis-induced intermittency. To begin with we find that the system possesses multiple attractors (including two chaotic ones) with fractal basin boundaries, over a wide range of control parameters. We also find parameter intervals over which the system has fractal parameter dependence for fixed initial conditions. Such fractal structures can give rise to a form of fragility (final-state sensitivity), whereby small changes in the initial state or the control parameters of the system can result in a different final outcome. We find parameter regions where as the control parameter is varied, the chaotic attractors merge into one attractor, thus resulting in crisis-induced intermittency. We verify this by investigating the phase space of the system and calculating the scaling exponent put forth by Grebogi *et al.* [2]. As far as we are aware, this is the first example of such behavior in a dynamo model as well as in a six-dimensional flow.

The structure of the paper is as follows. In Sec. II we briefly introduce the model. Section III summarizes our results demonstrating the presence of crisis in this model. Finally, Sec. IV contains our conclusions.

II. MODEL

The dynamo model considered here is the so-called $\alpha\omega$ mean-field dynamo model with a dynamic α effect given by Schmalz and Stix [16] (see also [17] for details). We assume a spherical axisymmetrical configuration with one spatial dimension x (measured in terms of the stellar radius R) for which the magnetic field takes the form

$$\vec{B}=\left(0,B_{\phi},\frac{1}{R}\frac{\partial A_{\phi}}{\partial x}\right), \tag{2.1}$$

where A_{ϕ} is the ϕ component (latitudinal) of the magnetic

*Electronic address: E.O.Covas@qmw.ac.uk
†Electronic address:reza@maths.qmw.ac.uk

vector potential and B_ϕ is the ϕ component of $\vec{B}$. The model is made up of two ingredients: (i) the mean-field induction equation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B} + \alpha \vec{B} - \eta_t \vec{\nabla} \times \vec{B}), \tag{2.2}$$

where $\vec{B}$ is the mean magnetic field, $\vec{v}$ is the mean velocity, η_t is the turbulent magnetic diffusitivity, and α represents the α effect, and (ii) the α effect, which arises from the correlation of small-scale turbulent velocity and magnetic fields [18] and is important in maintaining the dynamo action by relating the mean electrical current arising in helical turbulence to the mean magnetic field. Here α is assumed to be dynamic and expressible in the form $\alpha = \alpha_0 \cos x - \alpha_M(t)$, where α_0 is a constant and α_M is its dynamic part satisfying the equation

$$\frac{\partial \alpha_M}{\partial t} = \nu_t \frac{\partial^2 \alpha_M}{\partial x^2} + Q \vec{J} \cdot \vec{B}, \tag{2.3}$$

where Q is a physical constant, $\vec{J}$ is the electrical current, and ν_t is the turbulent diffusivity. These assumptions allow Eq. (2.2) to be split into the two equations

$$\frac{\partial A_\phi}{\partial t} = \frac{\eta_t}{R^2} \frac{\partial^2 A_\phi}{\partial x^2} + \alpha B_\phi, \tag{2.4}$$

$$\frac{\partial B_\phi}{\partial t} = \frac{\eta_t}{R^2} \frac{\partial^2 B_\phi}{\partial x^2} + \frac{\omega_0}{R} \frac{\partial A_\phi}{\partial x}. \tag{2.5}$$

Expressing these equations in a nondimensional form, relabeling the new variables thus

$$(A_\phi, B_\phi, \alpha_M) \Rightarrow (A, B, C), \tag{2.6}$$

and using a spectral expansion of the form

$$A = \sum_{n=1}^{N} A_n(t) \sin nx, \tag{2.7}$$

$$B = \sum_{n=1}^{N} B_n(t) \sin nx, \tag{2.8}$$

$$C = \sum_{n=1}^{N} C_n(t) \sin nx, \tag{2.9}$$

where N determines the truncation order, reduces Eqs. (2.3)–(2.5) into a set of ODEs, the dimension of which depends on the truncation order N. In [17] the models were taken to be antisymmetric with respect to the equator and it was found that the minimum truncation order N for which a similar asymptotic behavior existed was $N=4$. Here, in view of computational costs, we take this value of N for which the set of truncated equations becomes

$$\frac{\partial A_1}{\partial t} = -A_1 + \frac{DB_2}{2} - \frac{32B_2C_2}{15\pi} + \frac{64B_2C_4}{105\pi} + \frac{64B_4C_2}{105\pi} - \frac{128B_4C_4}{63\pi}, \tag{2.10}$$

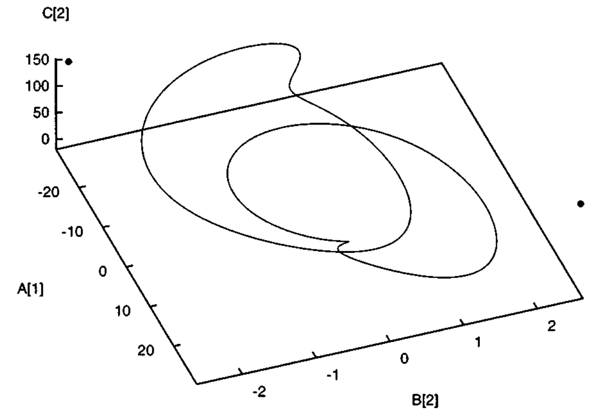

FIG. 1. Phase portraits of the two fixed points and the two stable cycles.

$$\frac{\partial B_2}{\partial t} = -4B_2 + \frac{8A_1}{3\pi} - \frac{24A_3}{5\pi}, \tag{2.11}$$

$$\frac{\partial C_2}{\partial t} = -4\nu C_2 + \frac{16A_1B_2}{5\pi} - \frac{32A_1B_4}{7\pi} + \frac{144A_3B_2}{7\pi} + \frac{416A_3B_4}{15\pi}, \tag{2.12}$$

$$\frac{\partial A_3}{\partial t} = -9A_3 + \frac{DB_2}{2} + \frac{DB_4}{2} - \frac{32B_2C_2}{21\pi} - \frac{64B_2C_4}{45\pi} - \frac{64B_4C_2}{45\pi} - \frac{128B_4C_4}{165\pi}, \tag{2.13}$$

$$\frac{\partial B_4}{\partial t} = -16B_4 + \frac{16A_1}{15\pi} + \frac{48A_3}{7\pi,} \tag{2.14}$$

$$\frac{\partial C_4}{\partial t} = -16\nu C_4 + \frac{96A_1B_2}{35\pi} + \frac{64A_1B_4}{21\pi} + \frac{32A_3B_2}{3\pi} + \frac{576A_3B_4}{55\pi}, \tag{2.15}$$

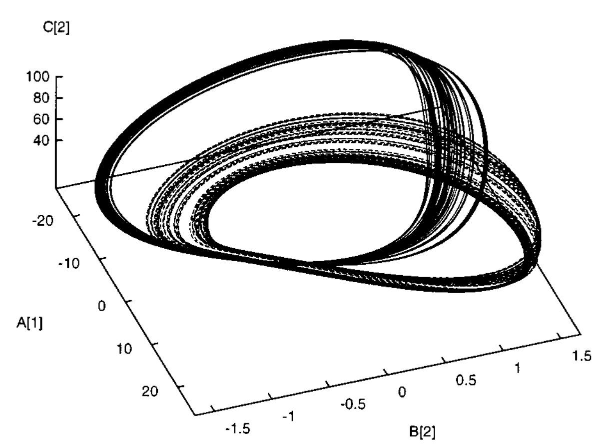

FIG. 2. Phase portraits of the two coexistent chaotic attractors.

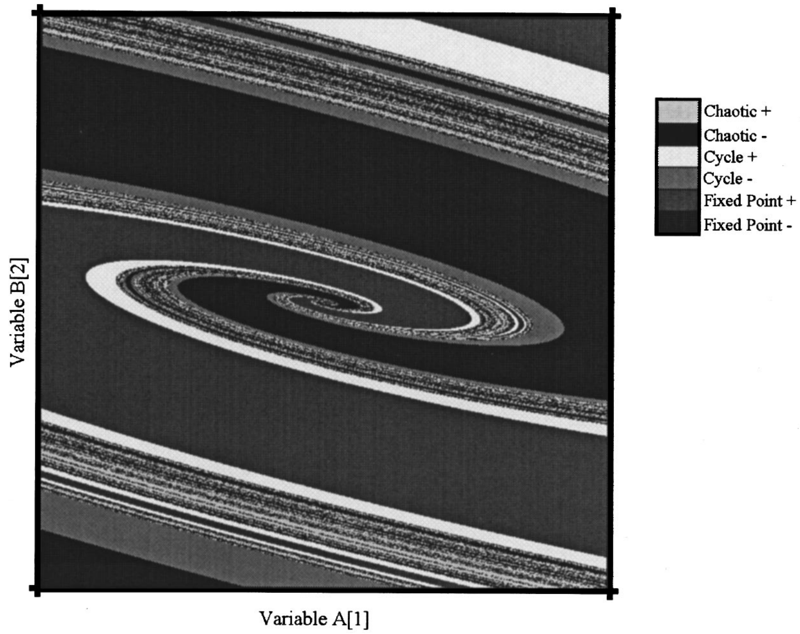

FIG. 3. A 800×800 grid showing a two-dimensional cut of the basins of attraction with $D=204$ and $C_2=A_3=B_4=C_4=0$. Variables A_1 and B_2 were centered at (0,0) and the size of the picture is 2×1. In the legend, + and − indicate the sign of the time average of A_1.

where D is the control parameter, the so-called dynamo number, and $\nu=\nu_t/\eta_t$, which for compatibility with [17,16] we take to be $\nu=0.5$.

Clearly the details of the resulting dynamics will depend on the truncation order chosen. For example, the $N=2$ case is expressible as the three-dimensional Lorenz system and the higher truncations can have different quantitative types of behavior. The important point, as far as our discussion here is concerned, is that the multiattractor regime discussed here seems to be present as the order of truncation is increased. In this way such a behavior might be of potential relevance in understanding some of the intermittent behavior in the output of the Sun and other stars.

III. CRISIS-INDUCED INTERMITTENCY

A coarse study of the system (2.10)–(2.15) and higher truncations was reported in [17] from a different point of view. Here we demonstrate the occurrence of crisis-induced intermittency in this system by considering the detailed nature of its attractors, their basins, and especially their metamorphoses (merging), while treating D as the control parameter.

To begin with we recall that symmetries are usually associated with this type of attractor merging. The six-dimensional dynamical system considered here possesses the symmetries

$$A_n \to -A_n, \quad B_n \to -B_n, \quad C_n \to C_n. \tag{3.1}$$

Now assuming the existence of a crisis for this system at $D=D_c$, then for crisis-induced intermittency to exist one requires that for $D<D_c$ there exist two (or more) chaotic attractors and that as D is increased the attractors enlarge and at $D=D_c$ they simultaneously touch the boundary separating their basins. In that case, for D slightly greater than D_c, a typical orbit will spend long periods of time in each of the regions where the attractors existed for $D<D_c$ and intermittently switch between them. An important signature for this mechanism is the way the average time τ between these switches scales with the system parameter D. According to Grebogi *et al.* [2], for a large class of dynamical systems, this relation takes the form

$$\tau \sim |D-D_c|^{-\gamma}, \tag{3.2}$$

where the real constant γ is the critical exponent characteristic of the system under consideration.

To show that crisis-induced intermittency occurs for the system (2.10)–(2.15), we begin by noting that our numerical results indicate that, for a wide range of parameter values, the system possesses multiple attractors consisting of fixed points, periodic orbits, and chaotic attractors. Starting around $D=195$, two cycles coexist and both bifurcate in a doubling bifurcation sequence into two chaotic attractors that coexist after $D>203$. At $D\approx 200.4$ two other periodic orbits appear that persist for the parameter values considered here. Figures 1 and 2 show these attractors for $D=204$, where all six co-

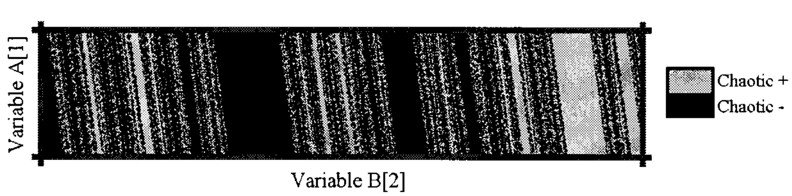

FIG. 4. A 800×160 grid showing the amplification of the previous picture (close to the lower left corner) with $A_1=-0.804$, $B_2=-0.700$, and size 0.01×0.002.

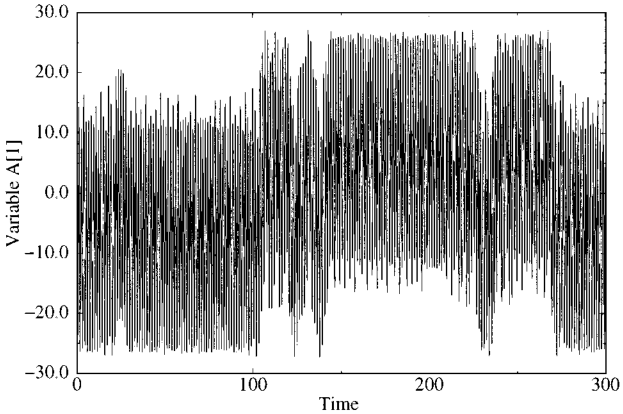

FIG. 5. Chaotic time series for the merged attractors for $D=205>D_c$.

exist and their positions in the six-dimensional phase are well separated (note that the apparent overlaps in Figs. 1 and 2 are due to projections).

We also found the corresponding basins of attraction for each attractor that indicate fractal boundaries. This can be seen in Fig. 3, which shows a two-dimensional cut $(C_2=A_3=B_4=C_4=0)$ of the basin boundary for this system at the parameter value $D=204$, and Fig. 4, which shows the magnification of a region of Fig. 3 where both chaotic attractors possess fractal basins [19]. We also calculated the box-counting dimension of the boundary between attractors on a horizontal one-dimensional cut of Fig. 4, which turned out to be noninteger, further substantiating the fractal nature of the boundaries.

Now as D is increased, the two chaotic attractors merge and give rise to a single connected attractor. Figure 5 shows the time series for the variable A_1 after the merging and Fig. 6 shows the projection of the merged attractors on the variables A_1, B_2, and C_2. Prior to $D_c \approx 204.2796$, there is no switch between the two attractors and the time series does not show the bimodal behavior seen in Fig. 5.

These results show a clear indication for the occurrence of crisis-induced intermittency in this model. To substantiate this further, we checked that for this system the scaling relation (3.2) is satisfied in the neighborhood of D_c

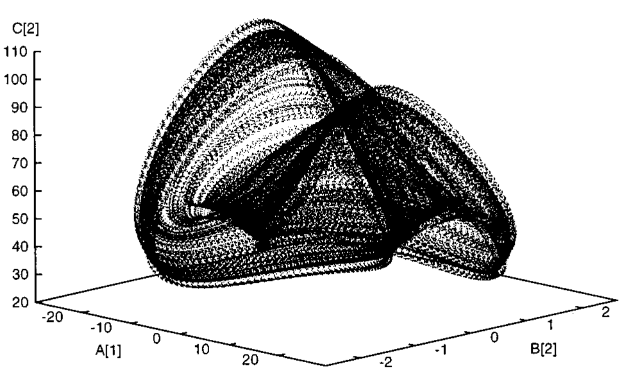

FIG. 6. Projection of the resulting merged chaotic attractor in the space A_1, B_2, C_2 for $D=207$.

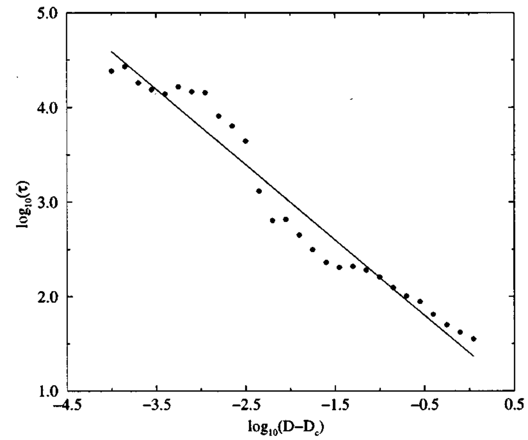

FIG. 7. Scaling of τ as a function of the distance to the critical dynamo number D_c together with the fitted line.

≈ 204.2796. Figure 7 shows the plot of $\log_{10}|\tau|$ versus $\log_{10}|D-D_c|$. To produce the plot, 28 points were taken at regular spacings with the initial conditions chosen in the chaotic basin of the merged attractor after $D \approx 204.2796$ and 200×10^6 iterations were taken for each point. The transitions between the ghosts of the previous attractors were detected using the averages of the variable A_1 over a pseudoperiod of approximately $\Delta t \approx 1.5$ nondimensional time units. As can be seen, the points are well approximated by a straight line, which was obtained using a least-squares fit giving $\gamma \approx 0.79 \pm 0.03$.

The γ coefficient can be calculated also from theoretical grounds, as shown by Grebogi *et al.* [2]. The method involves calculating the stable and unstable manifolds of the unstable orbit (thereafter B) mediating the crisis. By examining the trajectories around the transitions between the ghosts of the previous attractors at $D=204.35>D_c$, we found the point where the orbit went inside the portion of the unstable manifold of the B that has poked over to the other

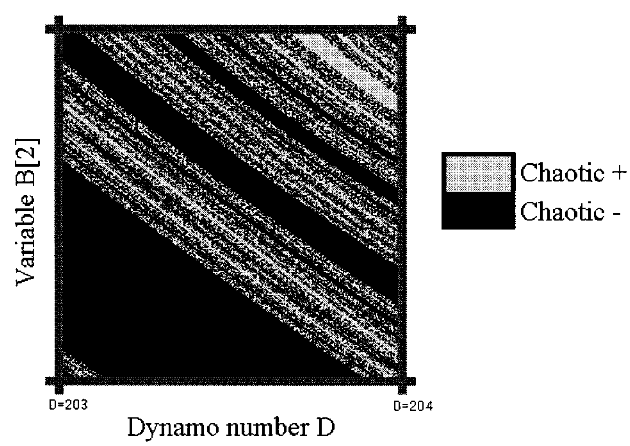

FIG. 8. Depiction of the final state (attractor) of the system as a function of changes in the parameter D and the initial condition B_2. This figure represents a horizontal slice of Fig. 4 for many runs with different dynamo numbers. A resolution of 300×300 pixels was used and all initial conditions were taken to be zero except for $A_1=-0.80$ and B_2 centered at -0.70.

side of the stable manifold of B. The orbit then follows closely the orientation of the stable and unstable manifolds. We then calculated an estimate of the direction of the unstable and stable manifolds. Since this was very sensitive, the value of γ had a large error bar, that is, the calculated value could be anywhere in the range [0.4,1.2], depending on minor changes in the choice of the vectors that determine the unstable and stable manifolds. Because the system was high dimensional, all the projections in two-dimensional planes we used were not very useful to determine with good precision the directions of the two manifolds. Therefore, we were unable to calculate the critical exponent with sufficient precision to compare with the one calculated from the time between flips of the orbit.

Finally, we looked at the parameter dependence of the system for fixed initial conditions. We found that there are intervals of D for which this is fractal. This can be seen from Fig. 8 which depicts the final state (attractor) of the system (2.10)–(2.15) as a function of changes in the parameter D and the initial condition B_2.

IV. CONCLUSION

We have found the presence of multiple attractors with fractal basin boundaries as well as crisis-induced intermittency in a truncated axisymmetric $\alpha\omega$ dynamo model that is antisymmetric with respect to the equator. We have seen that this type of intermittency is due to the collision of the two chaotic attractors and have confirmed this by calculating the scaling coefficient suggested by Grebogi *et al.* [2]. The presence of crisis-induced intermittency, coupled with the facts that this type of multiple attractors seem to persist in higher-order truncations and the presence of symmetry in dynamo models, may indicate the relevance of this type of intermittency in more realistic dynamo settings.

We have also found that this system possesses a fractal parameter dependence for fixed initial conditions. The presence of such fractal structures results in a form of fragility (final-state sensitivity), whereby small changes in the initial conditions or the control parameter of the system can result in qualitative changes in its final dynamics. This type of sensitivity could be of significance in astrophysics in that, for example, it could potentially lead to stars of same spectral type, rotational period, age, and compositions showing different modes of dynamical behavior [20]. Finally, as far as we are aware, this is the first instance of such behavior in a dynamo model as well as in a six-dimensional flow.

ACKNOWLEDGMENTS

We would like to thank John Brooke and Andrew Tworkowski for helpful discussions. E.C. is supported by Grant No. BD/5708/95, Program PRAXIS XXI, from JNICT, Portugal. R.T. benefited from PPARC, United Kingdom Grant No. H09454. This research also benefited from the EC Human Capital and Mobility (Networks) Grant No. ERBCHRXCT940483.

[1] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **48**, 1507 (1982).
[2] C. Grebogi, E. Ott, F. Romeiras, and J. A. Yorke, Phys. Rev. A **36**, 5365 (1987).
[3] W. L. Ditto, S. Rauseo, R. Cawley, C. Grebogi, G. H. Hsu, E. Kostelich, E. Ott, H. T. Savage, R. Seguan, M. L. Spano, and V. A. York, Phys. Rev. Lett. **63**, 923 (1989).
[4] Ch. Karakotsou and A. N. Anagnostopoulos, Physica D **93**, 157 (1986).
[5] E. Ott, *Chaos in Dynamic Systems* (Cambridge University Press, Cambridge, 1993).
[6] F. Feudel, W. Jansen, and J. Kurths, Int. J. Bifurcat. Chaos, **3**, 131 (1993).
[7] N. O. Weiss, in *Lectures on Solar and Planetary Dynamos* (Ref. [10]).
[8] N. O. Weiss, F. Cattaneo, and C. A. Jones, Geophys. Astrophys. Fluid Dyn. **30**, 305 (1984).
[9] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *Magnetic Fields in Astrophysics* (Gordon and Breach, New York, 1983).
[10] E. A. Spiegel, in *Lectures on Solar and Planetary Dynamos*, edited by M. R. E. Proctor and A. D. Gilbert (Cambridge University Press, Cambridge, 1994).
[11] D. Schmitt, M. Schüssler, and A. Ferriz-Mas, Astron. Astrophys. **311**, L1 (1996).
[12] S. M. Tobias, Astron. Astrophys. **307**, L21 (1996).
[13] S. M. Tobias, N. O. Weiss, and V. Kirk, Mon. Not. R. Astron. Soc. **273**, 1150 (1995).
[14] A. S. Tworkowski, R. K. Tavakol, A. Brandenburg, D. Moss, and I. Tuominen (unpublished).
[15] J. M. Brooke and D. Moss, Astron. Astrophys. **303**, 307 (1995).
[16] S. Schmalz and M. Stix, Astron. Astrophys. **245**, 654 (1991).
[17] E. Covas, A. Tworkowski, A. Brandenburg, and R. Tavakol, Astron. Astrophys. **317**, 610 (1997).
[18] F. Krause and K.-H. Rädler, *Mean Field Magnetohydrodynamics and Dynamo Theory* (Pergamon, Oxford, 1980).
[19] A version of this basin picture in full color is available at the following web address: *http://www.maths.qmw.ac.uk/~eoc/basin.html*
[20] R. K. Tavakol, A. S. Tworkowski, A. Brandenburg, D. Moss, and I. Tuominen, Astron. Astrophys. **296**, 269 (1995).